View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Elsevier - Publisher Connector

Topology and its Applications 37 (1990) 131-136
North-Holland

DEGREE-1 MAPS INTO LENS SPACES AND FREE CYCLIC ACTIONS ON HOMOLOGY 3-SPHERES*

E. LUFT and D. SJERVE

Department of Mathematics, The University of British Columbia, Vancouver, BC, Canada V6T 1Y4

Received 20 September 1988
Revised 7 August 1989

If M is a closed orientable 3-manifold with $H_1(M) = \mathbb{Z}_n$, then there is a lens space $L_{n,m}$ unique up to homotopy, and a degree-1 map $f: M \to L_{n,m}$. As a corollary we prove that if $\tilde{M}$ is a homology 3-sphere admitting a fixed point free action by $\mathbb{Z}_n$, then the regular covering $\tilde{M} \to \tilde{M}/\mathbb{Z}_n$ is induced from the universal covering $S^3 \to L_{n,m}$ by a degree-1 map $f: \tilde{M}/\mathbb{Z}_n \to L_{n,m}$.

AMS (MOS) Subj. Class.: Primary 57M99

homology 3-spheres   degree-1 maps   lens spaces

1. Introduction

In this paper we will be concerned with the existence of degree-1 maps $f: M \to L_{n,m}$ where M is a closed orientable 3-manifold and $L_{n,m}$ is some lens space. If $H_1(M) = \mathbb{Z}_n$, such maps exist and the lens space $L_{n,m}$ is unique up to homotopy equivalence (see Theorem 3.4). If $\tilde{M}$ is a homology 3-sphere and $p : \tilde{M} \to M$ is a regular covering with the cyclic group $\mathbb{Z}_n$ as its covering group, then $H_1(M) \cong \mathbb{Z}_n$ and the covering $p: \tilde{M} \to M$ is induced from the universal covering $q: S^3 \to L_{n,m}$ by means of a degree-1 map $f: M \to L_{n,m}$ (see Theorem 3.5). An explicit and complete description of Seifert fibered homology 3-spheres $\tilde{M}$ with fixed point free cyclic group actions is given in [5].

We would like to thank the referee for his comments.

2. Preliminaries

Throughout this paper we work in the PL category. A PL homeomorphism we simply call an isomorphism. Our reference for 3-manifold concepts is [3].

A homology 3-sphere is a 3-manifold with the same homology as a 3-sphere.

* Research partially supported by NSERC grants A3503 and A7218.

0166-8641/90/$03.50 © 1990—Elsevier Science Publishers B.V. (North-Holland)

A surface is a connected compact 2-manifold. A surface F in a 3-manifold W is proper if $F \cap \partial W = \partial F$. We call a 2-sided proper surface $F \subset W$ a Seifert surface if ∂F is a single 1-sphere that does not bound a 2-cell in ∂W.

If A is an Abelian group, $T(A)$ denotes its torsion subgroup. If X is a space with finitely generated homology, $\beta_i(X)$ denotes the rank of the ith homology group $H_i(X)$.

Proposition 2.1. *Let W be a compact 3-manifold with ∂W a torus. Suppose there is a 1-sphere $S^1 \subset \partial W$ such that $H_1(W) = \mathbb{Z}[S^1] \oplus A$. Then there is a Seifert surface $F \subset W$ such that ∂F and S^1 intersect transversally in exactly one point.*

Proof. Let $\phi : H_1(M) \to \mathbb{Z}$ be the epimorphism defined by $\phi[S^1] = 1$ and $\phi(A) = 0$. Then there is a map $f : W \to S^1$ such that

$$f_* = \phi : H_1(W) \to H_1(S^1) = \mathbb{Z}.$$

There is a 1-sphere $\Sigma^1 \subset \partial W$ such that $\partial W = S^1 \times \Sigma^1$ and $f_*[\Sigma^1] = 0$. Changing the map $f : \partial W \to S^1$ by a homotopy, we may assume that $f(x, y) = x$ and that f is transversal with respect to a point $x_0 \in S^1$. Let F be the component of $f^{-1}(x_0)$ with $\partial F \neq \emptyset$. □

Corollary 2.2. *Let W be an orientable compact 3-manifold with ∂W a torus and with $H_1(W) = \mathbb{Z}$. Then there is a Seifert surface $F \subset W$.*

Proof. From the Euler characteristic formula it follows that $0 = \frac{1}{2}\chi(\partial W) = \chi(W) = 1 - \beta_1(W) + \beta_2(W)$. Therefore $\beta_2(W) = 0$. From the universal-coefficient formula we conclude that $T(H^2(W)) = T(H_1(W)) = 0$. Therefore $H^2(W) = 0$. By Poincaré duality, $H_1(W, \partial W) = H^2(W) = 0$. From the exact homology sequence,

$$H_1(\partial W) \to H_1(W) = \mathbb{Z} \to H_1(W, \partial W) = 0,$$

we conclude that there is a 1-sphere $S^1 \subset \partial W$ such that $[S^1]$ is a generator of $H_1(W)$. By Proposition 2.1 there is a Seifert surface $F \subset W$. □

Proposition 2.3. *Let $p : \tilde{X} \to X$ be a regular covering projection and let G be its group of covering transformations. If $H_1(\tilde{X}) = 0$, then the natural epimorphism $\nu : \pi_1(X) \to G$ induces an isomorphism $\nu_* : H_1(X) \to H_1(G)$.*

Proof. Associated to the regular covering $p : \tilde{X} \to X$ is the short exact sequence

$$1 \to \pi_1(\tilde{X}) \xrightarrow{p_*} \pi_1(X) \xrightarrow{\nu} G \to 1$$

which then gives the exact sequence (see [4, p. 203])

$$\pi_1(\tilde{X})/[\pi_1(\tilde{X}), \pi_1(X)] \to H_1(\pi_1(X)) \xrightarrow{\nu_*} H_1(G) \to 0.$$

But $H_1(\tilde{X}) = \pi_1(\tilde{X})/[\pi_1(\tilde{X}), \pi_1(\tilde{X})] = 0$. □

3. Degree-1 maps onto lens spaces

The following lemmas will be applied.

Lemma 3.1. *Let M be a closed orientable 3-manifold with $H_1(M)=\mathbb{Z}_n$. Let $S^1\subset M$ be a 1-sphere such that $[S^1]$ is a generator of $H_1(M)$ and let $N=S^1\times D^2$ be a regular neighborhood of S^1 in M. Then $H_1(\overline{M-N})=\mathbb{Z}$ and there is a Seifert surface $F\subset\overline{M-N}$ with $[\partial F]=n[S^1]+m[\partial D^2]$ in $H_1(\partial N)$ for some integer m.*

Proof. Denote $W=\overline{M-N}$. From the exact sequence

$$H_1(N)=\mathbb{Z}\xrightarrow{\text{onto}}H_1(M)=\mathbb{Z}_n\to H_1(M,N)\to 0$$

it follows that $H_1(M,N)=0$ and by excision that $H_1(W,\partial W)=0$. From the Euler characteristic formula $\chi(W)=\frac{1}{2}\chi(\partial W)=0=1-\beta_1(W)+\beta_2(W)$ we obtain that $\beta_1(W)=1+\beta_2(W)$. By Poincaré duality $H_2(W)\cong H^1(W,\partial W)=0$. Therefore $\beta_1(W)=1$. By the universal-coefficient formula $T(H_1(W))=T(H^2(W))$. By Poincaré duality $H^2(W)=H_1(W,\partial W)=0$. We conclude that $H_1(W)=\mathbb{Z}$.

From the exact sequence

$$H_1(\partial W)=\mathbb{Z}\oplus\mathbb{Z}\to H_1(W)=\mathbb{Z}\to H_1(W,\partial W)=0$$

it follows that there is a 1-sphere $S_0^1\subset\partial W$ such that $[S_0^1]$ is a generator of $H_1(W)=\mathbb{Z}$. By Proposition 2.1 there is a Seifert surface $F\subset W$ intersecting S_0^1 transversally in exactly one point. Therefore $\partial W=\partial N=S_0^1\times\partial F$. From the Mayer–Vietoris sequence of the manifolds N and W,

$$0\to H_1(\partial N)=\mathbb{Z}\oplus\mathbb{Z}\to H_1(N)\oplus H_1(W)=\mathbb{Z}\oplus\mathbb{Z}\to H_1(M)=\mathbb{Z}_n\to 0,$$

it follows that $[\partial F]=n[S^1]+m[\partial D^2]$ in $H_1(\partial N)$. □

Lemma 3.2. *Let M be a closed orientable 3-manifold, $S^1\subset M$ a 1-sphere, $N=S^1\times D^2$ a regular neighborhood of S^1 in M, and $F\subset\overline{M-N}$ a Seifert surface with $[\partial F]=n[S^1]+m[\partial D^2]$ in $H_1(\partial N)$. Then*

(1) *F determines an element $[F]\in H_2(M,\mathbb{Z}_n)$;*

(2) *if $\beta: H_2(M,\mathbb{Z}_n)\to H_1(M,\mathbb{Z}_n)$ is the Bockstein homomorphism defined by the exact sequence $0\to\mathbb{Z}_n\xrightarrow{i}\mathbb{Z}_{n^2}\to\mathbb{Z}_n\to 0$, $i(k)=kn$, then $\beta[F]=[S^1]$;*

(3) *if $\langle\,,\rangle: H_1(M,\mathbb{Z}_n)\times H_2(M,\mathbb{Z}_n)\to\mathbb{Z}_n$ is the intersection pairing* mod n, *then*

$$\langle[S^1],[F]\rangle=\pm m.$$

Proof. (1) Via a fixed triangulation F defines an element F in the singular chain group $C_2(M)$ with ∂F homologous to $n[S^1]$ in $C_1(M)$. Hence F defines a cycle in the chain group $C_2(M)\otimes\mathbb{Z}_n$ of the chain complex of $M(\text{mod } n)$, and therefore defines up to sign a unique homology class $[F]$ in $H_2(M,\mathbb{Z}_n)$.

(2) Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_2(M)\otimes\mathbb{Z}_n & \longrightarrow & C_2(M)\otimes\mathbb{Z}_{n^2} & \longrightarrow & C_2(M)\otimes\mathbb{Z}_n & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \partial\otimes\mathrm{id}} & & \downarrow & & \\ 0 & \longrightarrow & C_1(M)\otimes\mathbb{Z}_n & \xrightarrow{\mathrm{id}\otimes i} & C_1(M)\otimes\mathbb{Z}_{n^2} & \longrightarrow & C_1(M)\otimes\mathbb{Z}_n & \longrightarrow & 0. \end{array}$$

F defines also an element $F\in C_2(M)\otimes\mathbb{Z}_{n^2}$. Note that $(\mathrm{id}\otimes i)S^1=(\partial\otimes\mathrm{id})F$. It follows from the definition of β that $\beta[F]=[S^1]$.

(3) Let $y_0\in\partial D^2$ and let $\hat{F}$ be the surface obtained from F by adding a collar contained in N to F. Then $S^1\times y_0$ intersects $\hat{F}$ transversally in exactly m points with the same sign. Therefore

$$\langle[S^1],[F]\rangle=\langle S^1\times y_0,\hat{F}\rangle=\pm m. \qquad \square$$

Lemma 3.3. *Let M be a closed orientable 3-manifold with $H_1(M,\mathbb{Z}_n)=\mathbb{Z}_n$. Let S^1, $S_0^1\subset M$ be 1-spheres with $[S^1]$, $[S_0^1]$ generators of $H_1(M)$, let $N=S^1\times D^2$, $N=S_0^1\times D_0^2$ be regular neighborhoods of S^1, S_0^1 in M respectively, and let $F\subset\overline{M-N}$, $F_0\subset\overline{M-N_0}$ be Seifert surfaces with $[\partial F]=n[S^1]+m[\partial D^2]$ in $H_1(\partial N)$ and $[\partial F_0]=n[S_0^1]+m_0[\partial D_0^2]$ in $H_1(\partial N_0)$. Suppose that $[S_0^1]=k[S^1]$ in $H_1(M)$. Then*

$$m_0=\pm mk^2 \quad \textit{in } \mathbb{Z}_n.$$

Proof. By the universal-coefficient formula $H_2(M,\mathbb{Z}_n)=\mathbb{Z}_n$. We apply Lemma 3.2. Since $\beta[F]=[S^1]$ is a generator of $H_1(M,\mathbb{Z}_n)$, it follows that $\beta: H_2(M,\mathbb{Z}_n)\to H_1(M,\mathbb{Z}_n)$ is an isomorphism. From $\beta(k[F])=k[S^1]=[S_0^1]=\beta[F_0]$ we conclude that $[F_0]=k[F]$. Therefore

$$\pm m_0=\langle[S_0^1],[F_0]\rangle=\langle k[S^1],k[F]\rangle=k^2\langle[S^1],[F]\rangle=\pm k^2m$$

in $\mathbb{Z}_n$. $\square$

The following theorem is our main result.

Theorem 3.4. *Let M be a closed orientable 3-manifold with $H_1(M)=\mathbb{Z}_n$. Then there is a lens space $L_{n,m}$ uniquely determined up to homotopy equivalence and a degree-1 map $f: M\to L_{n,m}$.*

Furthermore, the Bockstein homomorphism $\beta: H_2(M,\mathbb{Z}_n)\to H_1(M,\mathbb{Z}_n)$ defined by the exact sequence $0\to\mathbb{Z}_n\xrightarrow{i}\mathbb{Z}_{n^2}\to\mathbb{Z}_n\to 0$ is an isomorphism. If $x\in H_2(M,\mathbb{Z}_n)=\mathbb{Z}_n$ is a generator, then the mod n *intersection number $\langle\beta(x),x\rangle=\pm mk^2$ is a unit in $\mathbb{Z}_n$. It determines uniquely the lens space $L_{n,m}$ up to homotopy equivalence.*

Proof. Let $S^1\subset M$ be a 1-sphere such that $[S^1]$ is a generator of $H_1(M)=\mathbb{Z}_n$ and let $N=S^1\times D^2$ be a regular neighborhood of S^1 in M. By Lemma 3.1 there is a Seifert surface $F\subset\overline{M-N}$ with $[\partial F]=n[S^1]+m[\partial D^2]$ in $H_1(\partial N)$. Let $L_{n,m}$ be the lens space of type (n,m), i.e., $L_{n,m}=V\cup V_1$, $V=S^1\times D^2$, $V_1=S_1^1\times D_1^2$ are solid tori with $V \cap V_1 = \partial V = \partial V_1$ and $[\partial D_1^2] = n[S^1] + m[\partial D^2]$ in $H_1(\partial V)$. We construct a degree-1 map $f : M \to L_{n,m}$ as follows: First let $f| : N \to V$ be the identity map. Extend $f|_N$ to $f| : N \cup F \to V \cup D_1^2$ by mapping F onto D_1^2. Since V_1 cut open along D_1^2 is a 3-cell, the map $f|_{N \cup F}$ can be extended to a map $f : M \to L_{n,m}$ with $f(\overline{M - N}) = V_1$. Since $f| : N = f^{-1}(V) \to V$ is an isomorphism, $f : M \to L_{n,m}$ has degree 1.

To prove uniqueness suppose that $h : M \to L_{n,m_0}$ is a degree-1 map onto the lens space L_{n,m_0}. We have the Heegaard splitting $L_{n,m_0} = V_0 \cup V_2$, $V_0 = S_0^1 \times D_0^2$, $V_2 = S_2^1 \times D_2^2$ solid tori with $V_0 \cap V_2 = \partial V_0 = \partial V_2$ and $[\partial D_2^2] = n[S_0^1] + m_0[\partial D_2^2]$ in $H_1(\partial V_0)$. By [6, Theorem 2.1], the map h is homotopic to a map $g : M \to L_{n,m_0}$ such that $g^{-1}(V_0)$ is a solid torus and $g| : g^{-1}(V_0) \to V_0$ is an isomorphism. Let $g^{-1}(V_0) = S_0^1 \times D_0^2$. We may assume that $g| : M - g^{-1}(V_0) \to V_2$ is transversal with respect to a fixed $y_0 \times D_2^2$. Then the component F of $g^{-1}(y_0 \times D_2^2)$ with $\partial F \neq \emptyset$ is a Seifert surface in $\overline{M - g^{-1}(V)}$ with $[\partial F] = n[S_0^1] + m_0[\partial D_0^2]$ in $H_1(\partial g^{-1}(V))$. By Lemma 3.3, $m_0 = \pm mk^2 (\mathrm{mod}\ n)$ for some k. Thus $m_0 m = \pm (mk)^2 (\mathrm{mod}\ n)$. But this implies that L_{n,m_0} and $L_{n,m}$ are homotopy equivalent (see e.g. [1, p. 96]).

The statement regarding the Bockstein homomorphism follows from Lemma 3.2. □

If n is a prime, a map $f : M \to L_{n,m}$ of degree 1 was constructed by a homotopy theoretic argument in [2].

Note that if a 3-manifold is regularly covered by a homology 3-sphere, then it is necessarily orientable (the covering transformations must preserve orientation by the Lefschetz fixed point theorem).

Corollary 3.5. *Let $p : \tilde{M} \to M$ be a regular covering of the 3-manifold M by a homology 3-sphere $\tilde{M}$ and suppose that the group of covering transformations is the group $\mathbb{Z}_n$. Then there is a lens space $L_{n,m}$ uniquely determined up to homotopy equivalence and a degree-1 map $f : M \to L_{n,m}$ such that the regular covering $p : \tilde{M} \to M$ is induced from the standard regular covering $q : S^3 \to L_{n,m}$ by the degree-1 map $f : M \to L_{n,m}$.*

Proof. By Proposition 2.3, $H_1(M) = \mathbb{Z}_n$. By Theorem 3.4 there is a lens space $L_{n,m}$ uniquely determined up to homotopy equivalence and a degree-1 map $f : M \to L_{n,m}$. Therefore $f_* \pi_1(M) = \pi_1(L_{n,m})$ and $f_* : H_1(M) \to H_1(L_{n,m})$ must be an epimorphism and hence an isomorphism. It follows that $p_* \pi_1(\tilde{M}) = \ker(f_* : \pi_1(M) \to \pi_1(L_{n,m}))$ and hence that the regular covering $p : \tilde{M} \to M$ is induced from the standard regular covering $q : S^3 \to L_{n,m}$ by the map $f : M \to L_{n,m}$. □

References

[1] M. Cohen, A course in simple-homotopy theory, Graduate Texts in Mathematics 10 (Springer, New York, 1973).

[2] A. Edmonds, Construction of group actions on four-manifolds, Trans. Amer. Math. Soc. 299 (1987) 155–170.

[3] J. Hempel, 3-manifolds, Annals of Mathematics Studies 86 (Princeton Univ. Press, Princeton, NJ, 1976).
[4] P. Hilton and U. Stammbach, A course in homological algebra, Graduate Texts in Mathematics 4 (Springer, New York, 1970).
[5] E. Luft and D. Sjerve, On regular coverings of 3-manifolds by homology 3-spheres, to appear.
[6] F. Waldhausen, On mappings of handlebodies and of Heegaard splittings, in: Proceeding of the University of Georgia Top. of Manif. Institute (Markham, Chicago, 1969).